Girraween High School

2023

TRIAL HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 2

Total Marks: 100

Section 1 (Pages 2 – 5) **10 Marks**

- Attempt Q1 - Q10
- Allow about 15 minutes for this section

Section 2 (Pages 6-14) **90 marks**

- Attempt Q11 - Q16
- Allow about 2 hours and 45 minutes for this section

General Instructions

- Reading time: 10 minutes
- Working time: 3 Hours
- Write using a black or blue pen
- Board approved calculators may be used
- Laminated reference sheets are provided
- Answer multiple-choice questions by completely colouring in the appropriate circle on your multiple-choice answer sheet.
- Answer questions 11-16 in the appropriate answer booklet and show all relevant mathematical reasoning and/or calculations.

Section 1 (10 marks) Multiple choice

Attempt Questions 1-10

Allow about 15 minutes for this section

Question 1

$i^{2023} =$

(A) i **(B)** -1 **(C)** $-i$ **(D)** -1

Question 2

A cubic polynomial $P(x)$ with all real coefficients has roots α, β and γ such that $\alpha^2 + \beta^2 + \gamma^2 < 0$. The equation $P(x) = 0$ has

(A) No real root. **(B)** One real root.

(C) Two real roots. **(D)** Three real roots.

Question 3

One of the roots of the polynomial equation $2z^3 + 7z^2 + 22z - 13 = 0$ is $z = -2 + 3i$. The other two roots are

(A) $z = -2 - 3i \text{ and } z = \frac{1}{2}$ **(B)** $z = -2 - 3i \text{ and } z = -\frac{1}{2}$

(C) $z = 2 + 3i \text{ and } z = \frac{1}{2}$ **(D)** $z = 2 - 3i \text{ and } z = -\frac{1}{2}$

Question 4

Given that $\underset{\sim}{p}$ and $\underset{\sim}{q}$ are non-zero vectors, the contrapositive of:

$if\ \underset{\sim}{p}.\underset{\sim}{q} = 0\ then\ \underset{\sim}{p} \perp \underset{\sim}{q}$ is

(A) If $\underset{\sim}{p} \perp \underset{\sim}{q}$ then $\underset{\sim}{p}.\underset{\sim}{q} = 0$ **(B)** If $\underset{\sim}{p}$ is NOT $\perp$ $\underset{\sim}{q}$ then $\underset{\sim}{p}.\underset{\sim}{q} \neq 0$

(C) If $\underset{\sim}{p} \perp \underset{\sim}{q}$ then $\underset{\sim}{p}.\underset{\sim}{q} \neq 0$ **(D)** If $\underset{\sim}{p}$ is NOT $\perp$ $\underset{\sim}{q}$ then $\underset{\sim}{p}.\underset{\sim}{q} = 0$

Multiple choice continues on the following page

Question 5

Given that $\underset{\sim}{p}$ and $\underset{\sim}{q}$ are non-zero vectors, the converse of:

$if\ \underset{\sim}{p}.\underset{\sim}{q} = 0\ then\ \underset{\sim}{p} \perp \underset{\sim}{q}$ is

(A) If $\underset{\sim}{p} \perp \underset{\sim}{q}$ then $\underset{\sim}{p}.\underset{\sim}{q} = 0$

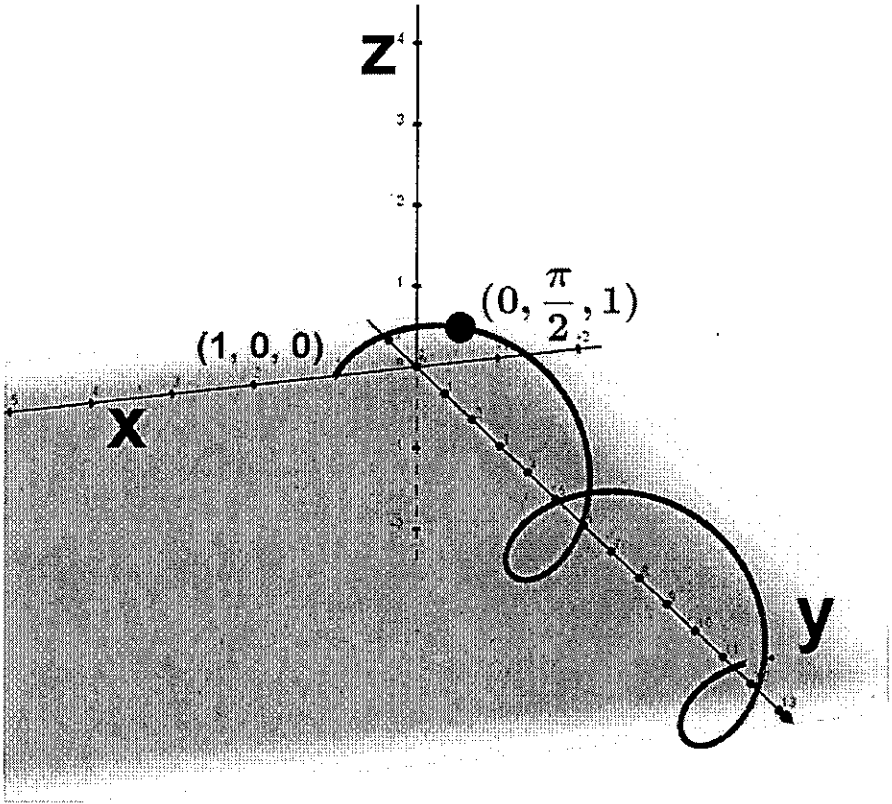
(B) If $\underset{\sim}{p}$ is NOT $\perp$ $\underset{\sim}{q}$ then $\underset{\sim}{p}.\underset{\sim}{q} \neq 0$

(C) If $\underset{\sim}{p} \perp \underset{\sim}{q}$ then $\underset{\sim}{p}.\underset{\sim}{q} \neq 0$

(D) If $\underset{\sim}{p}$ is NOT $\perp$ $\underset{\sim}{q}$ then $\underset{\sim}{p}.\underset{\sim}{q} = 0$

Question 6

A curve in three dimensional space is pictured below.



The equation for this curve is

(A) $(\cos t, \sin t, t)$

(B) $(\cos t, t, \sin t)$

(C) $(t, \cos t, \sin t)$

(D) $(\sin t, t, \cos t)$

Multiple choice continues on the following page

Question 7

A particle moving with simple harmonic motion (SHM) starts from rest at $x = 1$.

It next stops at $x = 9$. It next returns to $x = 1$ π seconds later. A possible equation

For the displacement of this particle at time t is

(A) $x = 4\sin 2t + 5$ **(B)** $x = 4\cos 2t + 5$

(C) $x = -4\sin 2t + 5$ **(D)** $x = -4\cos 2t + 5$

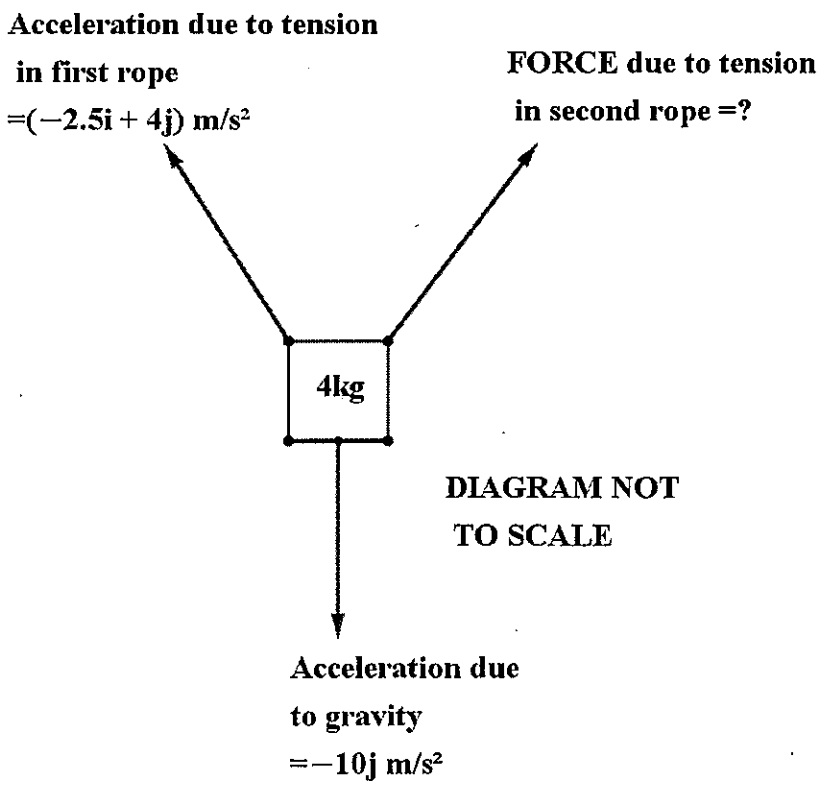
Question 8

A particle with mass $4kg$ is hanging stationary from two taut ropes. The *acceleration* due to tension in the first rope (with direction) is $(-2.5\underset{\sim}{i} + 4\underset{\sim}{j})\ m/s^2$. If the acceleration due to gravity is $(-10\underset{\sim}{j})m/s^2$ (see diagram), the total FORCE in the second rope is

(A) 10 Newtons **(B)** 12 Newtons

(C) 13 Newtons **(D)** 26 Newtons

Diagram for Q8:



Multiple choice continues on the following page

Question 9

$$\int_0^a f(2a-x).dx =$$

(A) $\int_a^{2a} f(x).dx$

(B) $\int_0^a f(x).dx$

(C) $\int_{2a}^a f(x).dx$

(D) $\int_a^0 f(x).dx$

Question 10

$$\int x^n \ln(x).dx =$$

(A) $\frac{x^{n+1}}{n+1} - \int x^n \ln x.dx$

(B) $x^{n-1} - \int \frac{x^{n+1}}{n+1} \ln x.dx$

(C) $x^n lnx - \int \frac{x^n}{n+1}.dx$

(D) $\frac{x^{n+1}}{n+1} lnx - \int \frac{x^n}{n+1}.dx$

Examination continues on the following page

Section II (90 marks)

Attempt Questions 11-16

Allow about 2 hours and 45 minutes for this section

Start the answers to each question on a separate page in your answer booklet.

In Questions 11-16 your responses should include all relevant mathematical reasoning and/ or calculations. *(Note: for complex number questions, the notation cis is acceptable).*

Question 11 ***(15 marks)*** **Marks**

(a) (i) Find $\frac{1+i\sqrt{3}}{1+i}$. Leave your answer in Cartesian form. **2**

(ii) Find $\frac{1+i\sqrt{3}}{1+i}$ in modulus/argument form. **2**

(iii) Hence find the exact value of $\cos\frac{\pi}{12}$. **1**

(b) (i) By letting $\sqrt{7+24i} = x + iy,\ x,\ y\ real,$ find $\sqrt{7+24i}$ in Cartesian form. **3**

(ii) Hence solve $z^2 + (4-i)z + (2-8i) = 0$. **2**

(c) Sketch and shade the region in the complex plane where $|z-1-i| \geq 2$ and $\frac{\pi}{4} \leq Arg\ z \leq \frac{\pi}{2}$ hold simultaneously. **3**

(d) Use DeMoivre's Theorem to find all of the 7th roots of $128i$. Leave your answers in modulus/argument form. **2**

Examination continues on the following page

Question 12 ***(15 marks)*** **Marks**

(a) Find $\int \frac{1}{2-\cos x}\,.dx$ **3**

(b) **(i)** Express $\frac{-16x-64}{(x+3)^2(x-1)}$ in the form $\frac{A}{(x+3)^2}+\frac{B}{(x+3)}+\frac{C}{(x-1)}$. **3**

(ii) Hence find $\int \frac{-16x-64}{(x+3)^2(x-1)}\,.dx$ **1**

(c) Find $\int x\cos x.\,dx$ **2**

(d) A particle moves so that its position at time t is given by

$x = 5\cos 2t - 5\sqrt{3}\sin 2t$.

(i) Prove the particle is moving in Simple Harmonic Motion. **1**

(ii) Express the particle's motion in the form **2**

$x = A\cos(2t+\alpha)$.

(iii) Find the period and the amplitude of the particle's motion **3**

and its initial position.

Examination continues on the following page

Question 13 **Marks**

(a) (i) Show BY FINDING IT (NOT BY SUBSTITUTION) that the point of intersection of $\begin{pmatrix} -9 \\ -14 \\ -3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 23 \\ -4 \\ -27 \end{pmatrix} + \lambda_2 \begin{pmatrix} 10 \\ 1 \\ -16 \end{pmatrix}$ is $\begin{pmatrix} 3 \\ -6 \\ 5 \end{pmatrix}$. **4**

(ii) Show that $\begin{pmatrix} -9 \\ -14 \\ -3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$ is a tangent to the sphere $(x-1)^2 + (y+2)^2 + (z-4)^2 = 21$ and show that the point of intersection of this line with the sphere is also $\begin{pmatrix} 3 \\ -6 \\ 5 \end{pmatrix}$. **2**

(iii) Find the equation of the straight line perpendicular to both $\begin{pmatrix} -9 \\ -14 \\ -3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 23 \\ -4 \\ -27 \end{pmatrix} + \lambda_2 \begin{pmatrix} 10 \\ 1 \\ -16 \end{pmatrix}$ passing through $\begin{pmatrix} 3 \\ -6 \\ 5 \end{pmatrix}$. **3**

Hence or otherwise show that $\begin{pmatrix} 23 \\ -4 \\ -27 \end{pmatrix} + \lambda_2 \begin{pmatrix} 10 \\ 1 \\ -16 \end{pmatrix}$ is also a tangent to the sphere $(x-1)^2 + (y+2)^2 + (z-4)^2 = 21$.

(b) Prove by mathematical induction that $7^n > 5^n + 2^n \ \forall\, n \in Z^+ \geq 2$. **4**

(c) Prove by contradiction that $\log_7 19$ is irrational. **2**

Examination continues on the following page

Question 14 *(15 marks)* **Marks**

(a) **(i)** Prove $a^2 + b^2 \geq 2ab \; \forall \; real \; a, b.$ **1**

(ii) Prove $a^2 + b^2 + c^2 \geq ab + ac + bc \; \forall \; real \; a, b, c.$ **1**

(iii) If a, b, c are the sides of a triangle, prove $c^2 > (a - b)^2$. **1**

(Be careful!)

(iv) Hence or otherwise prove that if a, b, c are the sides of a triangle, **1**

$a^2 + b^2 + c^2 < 2(ab + ac + bc).$

(b) **(i)** If $I_n = \int_0^{\frac{\pi}{4}} sin^n x \, . dx$, show that $I_n = \dfrac{-\left(\frac{1}{\sqrt{2}}\right)^n}{n} + \frac{n-1}{n} I_{n-2}$. **2**

(ii) Hence find $\int_0^{\frac{\pi}{4}} sin^5 x \, . dx$ **2**

(c) Find $\int \sqrt{\frac{x}{2-x}} \, . dx$ **3**

(d) **(i)** Use Euler's Theorem to prove that $2i \sin n\theta = e^{ni\theta} - e^{-ni\theta}$. **2**

(ii) Hence show that $sin^5\theta = \frac{sin5\theta}{16} - \frac{5 \sin 3\theta}{16} + \frac{5 \sin \theta}{8}$. **2**

__Examination continues on the following page__

Question 15 *(15 marks)* **Marks**

(a) A piston is moving up and down with Simple Harmonic Motion.

It weighs 1.2 kg.

At the top of its motion, the piston is 0.25m above the engine base.

At the bottom of its motion it is 0.15m above the engine base.

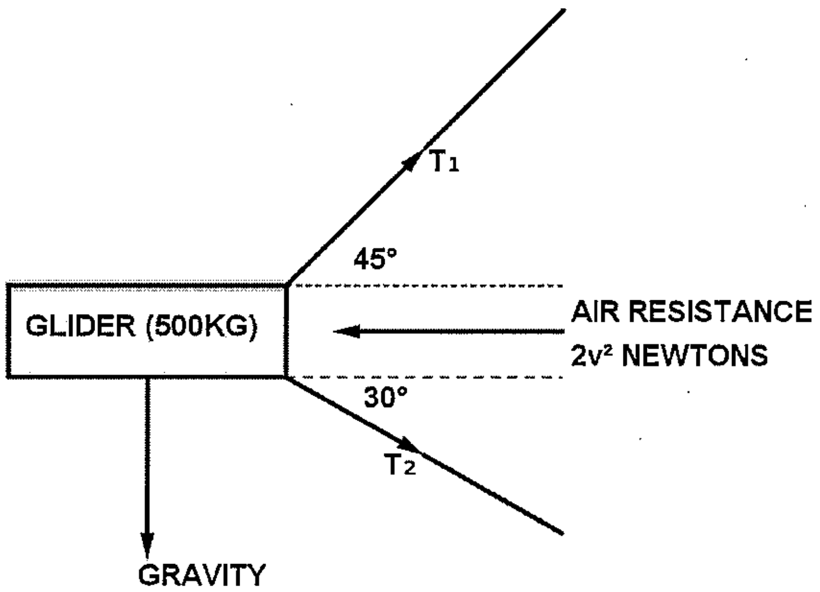
The piston completes 100 complete oscillations per second.

(i) Use the information above to write the piston's acceleration in the form $\ddot{x} = -n^2(x - C)$. **2**

(ii) Find the MAXIMUM force the piston exerts in Newtons. **2**

Question 15 continues on the following page

(b) A 500kg glider is being pulled behind an aeroplane which is flying horizontally. It is attached to the aeroplane by two ropes at angles of 45^o above and 30^o below the horizontal and the tensions in the two ropes are labelled T_1 and T_2 respectively. It experiences air resistance of $2v^2$ Newtons. The acceleration due to gravity is $10m/s^2$ *(see diagram below).*



(i) By resolving forces vertically, show that $T_2 = T_1\sqrt{2} - 10\ 000$. **2**

(ii) Given $T_1 = 10\ 000$ Newtons, by resolving forces horizontally find the total resultant force on the glider. **3**

Hence show that $\ddot{x} = 10(\sqrt{2} + \sqrt{6} - \sqrt{3}) - \frac{v^2}{250}$ and find the glider's limiting speed (the speed it approaches but can't reach) in metres per second.

(iii) Assuming the tensions in the ropes are constant and that the glider's initial speed is $50m/s$, find how far the glider travels before it reaches $70m/s$. **3**

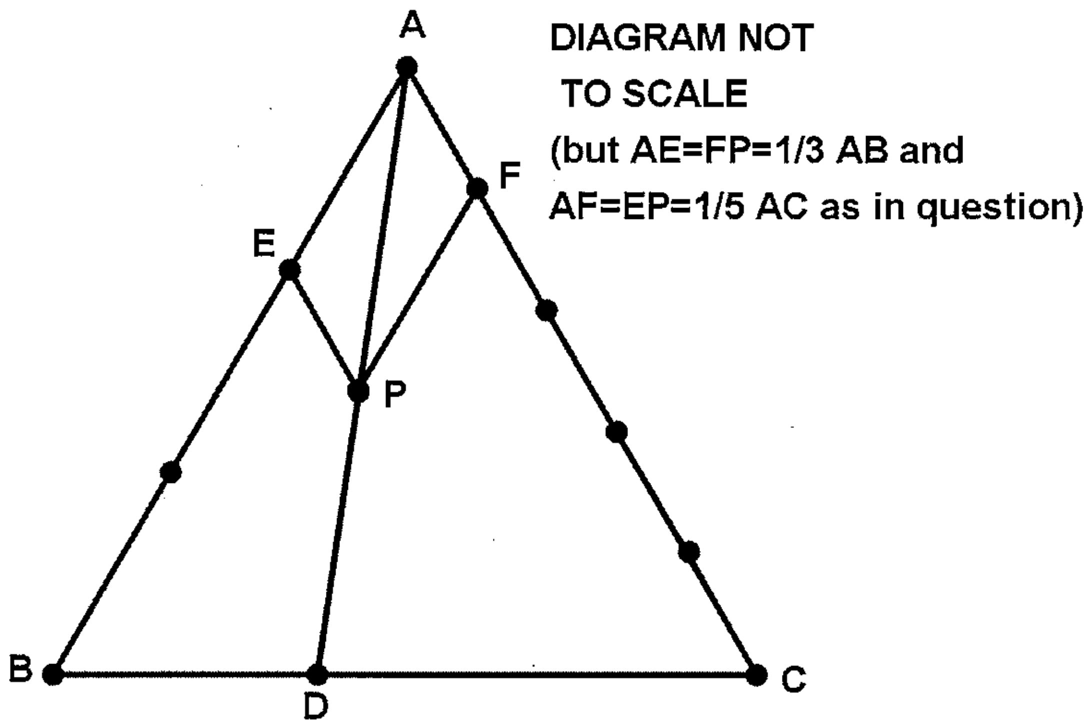
Question 15 continues on the following page

(c) ABC is an equilateral triangle. $\overrightarrow{AB} = \underset{\sim}{p}$ and *and* $\overrightarrow{AC} = \underset{\sim}{q}$.

$\overrightarrow{AE} = \overrightarrow{FP} = \frac{1}{3}\ \underset{\sim}{p}$ and $\overrightarrow{AF} = \overrightarrow{EP} = \frac{1}{5}\ \underset{\sim}{q}$. D is on BC so that

$\overrightarrow{BD} = \lambda\ \overrightarrow{BC}$. *(So note that D is NOT the midpoint of BC).*

(See diagram).



(i) Show that $\overrightarrow{BD} = \lambda(\ \underset{\sim}{q} - \underset{\sim}{p})$ and $AD = \underset{\sim}{p} + \lambda(\ \underset{\sim}{q} - \underset{\sim}{p}\ )$ **1**

(ii) Find the value of λ. **2**

Examination continues on the following page

Question 16 *(15 marks)* **Marks**

(a) **(i)** Show using DeMoivre's Theorem that **2**

$\sin 7\theta = 7\sin\theta - 56\,sin^3\,\theta + 112\,sin^5\theta - 64\,sin^7\theta.$

(ii) Hence solve the equation

$64x^7 - 112x^5 + 56x^3 - 7x + 1 = 0$. Leave your answers **2**

in the form $\sin\theta$, where $0 \leq \theta \leq 2\pi$.

(iii) Hence or otherwise, show that $\sin\frac{\pi}{14}\sin\frac{5\pi}{14}\sin\frac{17\pi}{14} = -\frac{1}{8}$. **2**

(b) A projectile with mass M is launched vertically into the air with initial speed $v_0 m/s$. It experiences air resistance directly against its motion of $0.2Mv$ Newtons, where v is the current velocity of the projectile. The ACCELERATION due to gravity is $10m/s^2$. Initially when the particle is launched the ground is at $x = 0$.

(i) While the projectile is ascending, show that the time since it **2**
was launched is given by $t = -5\ln\left(\frac{50+v}{50+v_0}\right)$ and find the time taken to reach the maximum height in terms of v_0.

(ii) Show that $x = -5(50 + v_0)e^{-0.2t} - 50t + 5(50 + v_0)$ and **3**
find an expression for the maximum height reached in terms of v_0.

(iii) The particle now starts to fall. It still experiences resistance **2**
AGAINST its direction of motion of $0.2Mv$ Newtons and acceleration due to gravity of $10m/s^2$. Now letting where the projectile starts to fall be $x = 0$, show that $x = 250e^{-0.2t} + 50t - 250$.

(iv) If the particle lands back on the ground where it was launched from **2**
10 seconds after it was launched, find the initial velocity (v_0) it was launched at.

END OF EXAMINATION!!!

GIRRAWEEN HIGH SCHOOL

MATHEMATICS EXTENSION 2
2023 TRIAL HIGHER SCHOOL CERTIFICATE

FINAL
Solutions

Student Number: ______________________________

This Booklet contains the answer sheet for Section 1 and Writing Booklet for Section 2.

Section 1 ANSWER SHEET

Select the alternative A, B, C or D that best answers the question.

1.	A	○	B	○	C	●	D	○
2.	A	○	B	●	C	○	D	○
3.	A	●	B	○	C	○	D	○
4.	A	○	B	●	C	○	D	○
5.	A	●	B	○	C	○	D	○
6.	A	○	B	●	C	○	D	○
7.	A	○	B	○	C	○	D	●
8.	A	○	B	○	C	○	D	●
9.	A	●	B	○	C	○	D	○
10.	A	○	B	○	C	○	D	●

Instructions

- If you need more paper for Section 2, please ask your supervisor.
- Write your student number on every booklet you use.
- Write on both sides of each sheet of paper.

Total number of booklets used ____________.

Multiple Choice:

(1) i^{2023}

$= i^{2020} \times i^{3}$

$= 1 \quad \times -i$

$= -i$ (C)

(2) If $\alpha^2+\beta^2+\gamma^2<0$

at least 1 of α, β, γ is not real.

As $P(x)$ has all real co-efficients, the conjugate of the non-real root must also be a root.

$\therefore P(x)=0$ has 2 non-real roots & 1 real root. (B)

(3) Like in Q.2, conjugate of $-2+3i$ [$=-2-3i$] is also a root.

By $\alpha+\beta+\gamma = -\frac{b}{a}$

$-2+3i+-2-3i+\gamma = -\frac{7}{2}$

$-4+\gamma = -\frac{7}{2}$

$\gamma = \frac{1}{2}$ (A)

Roots $= -2 \pm 3i, \frac{1}{2}$

(4) Contrapositive:

If $\underset{\sim}{p}$ is NOT $\perp \underset{\sim}{q}$, $\underset{\sim}{p}\cdot\underset{\sim}{q} \neq 0$ (B)

(5) Converse: If $\underset{\sim}{p} \perp \underset{\sim}{q}$, $\underset{\sim}{p}\cdot\underset{\sim}{q}=0$

(A)

(6) $t=y=0 \quad x=\cos 0=1 \quad z=\sin 0=0$

$(1,0,0)$.

$t=y=\frac{\pi}{2}$. $x=\cos\frac{\pi}{2}=0$. $y=\sin\frac{\pi}{2}=1$.

$(0,\frac{\pi}{2},1)$.

Parametric equation is

$(\cos t, t, \sin t)$ (B)

(7) Starts at LH end $\rightarrow -\cos$.

Period: $\frac{2\pi}{n}=\pi \Rightarrow n=2$.

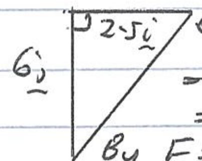
Centre of motion: $x=5$. Amplitude $=4$.

$x=-4\cos 2t+5$ (D)

(8) Total acceleration of 1^{ST} rope & gravity $=-2.5\underset{\sim}{i}-6\underset{\sim}{j}$

$\therefore$ Total acceleration of 2^{ND} rope

$=2.5\underset{\sim}{i}+6\underset{\sim}{j}$.



$\leftarrow$ Magnitude of acceleration

$=\sqrt{2.5^2+6^2}$

$=6.5$.

By $F=ma$, force $=6.5\times 4=26\,N$ (D)

(9) Let $u=2a-x$. $du=-1.dx$

$\int_0^a f(2a-x).dx$

$=\int_a^0 f(2a-x).-1.dx$

$=\int_{2a-a}^{2a-0} f(u).du$

$=\int_a^{2a} f(u).du=\int_a^{2a} f(x).dx$ (A)

(10) By $\int uv'.dx$

$=uv-\int vu'.dx$

$u=\ln x \quad v=\frac{x^{n+1}}{n+1}$

$u'=\frac{1}{x} \quad v'=x^n$

$\int x^n \ln x.dx=\frac{x^{n+1}}{n+1}\ln x-\int\frac{x^n}{n+1}.dx$ (D)

Q.(11)(a)(i) $\dfrac{1+i\sqrt{3}}{1+i}\times\dfrac{(1-i)}{(1-i)}$

$=\frac{1}{2}\left[(1+\sqrt{3})+i(\sqrt{3}-1)\right]$

(ii) $\dfrac{1+i\sqrt{3}}{1+i}$

$=\dfrac{2\operatorname{cis}\left[\frac{\pi}{3}\right]}{\sqrt{2}\operatorname{cis}\left[\frac{\pi}{4}\right]}$

$=\sqrt{2}\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$

(iii) Hence, equating reals

$\sqrt{2}\cos\frac{\pi}{12}=\frac{1+\sqrt{3}}{2}$

$\cos\frac{\pi}{12}=\frac{1+\sqrt{3}}{2\sqrt{2}}$

or $\cos\frac{\pi}{12}=\frac{\sqrt{2}+\sqrt{6}}{4}$

(b)(i) $x+iy=\sqrt{7+24i}$, x,y real.

$\therefore (x+iy)^2=7+24i$

$(x^2-y^2)+2ixy=7+24i$

Equating reals, $x^2-y^2=7$ (1)

Equating imaginaries $2xy=24$

$y=\frac{12}{x}$ (2)

Sub. (2) in (1):

$x^2-\frac{144}{x^2}=7$

$x^4-7x^2-144=0$

$(x^2-16)(x^2+9)=0$

$x=\pm4$ as x real, $x\neq\pm3i$.

$y=\frac{12}{x}=\pm3$

$\sqrt{7+24i}=\pm(4+3i)$

(ii) Solving $z^2+(4-i)z+(2-8i)=0$

$z=\dfrac{-(4-i)\pm\sqrt{(4-i)^2-4\times1\times(2-8i)}}{2\times1}$

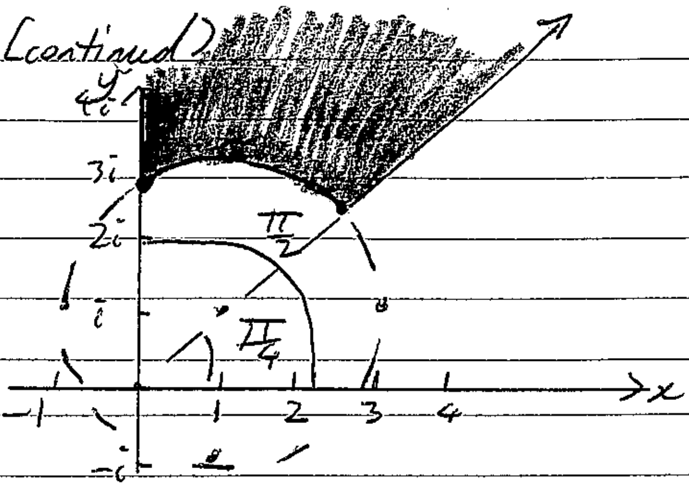
$z=\dfrac{-4+i-\sqrt{7+24i}}{2}$ or $z=\dfrac{-4+i+\sqrt{7+24i}}{2}$

$z=\dfrac{-4+i-4-3i}{2}$ or $z=\dfrac{-4+i+4+3i}{2}$

$z=-4-i$ or $z=2i$.

Q.(11) (continued)

(c)



(d) $128i = 128 \operatorname{cis} \frac{\pi}{2}$.

$\therefore \sqrt[7]{128i}$ by DeMoivre

$= 2\operatorname{cis}\left[\frac{\pi}{14} + \frac{2k\pi}{7}, k = 0, 1, 2, 3, 4, 5, 6.\right]$

$z = 2\operatorname{cis}\left[\frac{\pi}{14}\right], 2\operatorname{cis}\left[\frac{5\pi}{14}\right], 2\operatorname{cis}\left[\frac{9\pi}{14}\right], 2\operatorname{cis}\left[\frac{13\pi}{14}\right],$
$2\operatorname{cis}\frac{17\pi}{14}, 2\operatorname{cis}\frac{3\pi}{2}, 2\operatorname{cis}\frac{25\pi}{14}.$

or using principal arguments,

$z = 2\operatorname{cis}\left[\frac{\pi}{14}\right], 2\operatorname{cis}\left[\frac{5\pi}{14}\right], 2\operatorname{cis}\left[\frac{9\pi}{14}\right], 2\operatorname{cis}\left[\frac{13\pi}{14}\right],$
$2\operatorname{cis}\left[\frac{-11\pi}{14}\right], 2\operatorname{cis}\left[\frac{-\pi}{2}\right], 2\operatorname{cis}\left[\left(-\frac{3\pi}{14}\right)\right].$
$[= -2i]$

Solutions: GHS Trial p. 4

Q(12)(a) $\displaystyle\int \frac{1}{2-\cos x}\,dx$

$t=\tan\left(\frac{x}{2}\right)\quad \frac{dt}{dx}=\frac{\sec^2\left(\frac{x}{2}\right)}{2}=\frac{1+t^2}{2}.$

$dx=\frac{dx}{dt}\cdot dt=\frac{2}{1+t^2}\cdot dt$

$$=\int \frac{1}{2-\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt$$

$$=\int \frac{2}{1+3t^2}\,dt$$

$$=\frac{2}{3}\int \frac{1}{\frac{1}{3}+t^2}\,dt$$

$$=\frac{2\sqrt{3}}{3}\tan^{-1}(t\sqrt{3})+C$$

$$=\frac{2\sqrt{3}}{3}\tan^{-1}\left[\sqrt{3}\tan\left(\frac{x}{2}\right)\right]+C.$$

(b)(i) $\displaystyle\frac{-16x-64}{(x+3)^2(x-1)}=\frac{A}{(x+3)^2}+\frac{B}{(x+3)}+\frac{C}{(x-1)}$

$\therefore -16x-64=A(x-1)+B(x+3)(x-1)+C(x+3)^2$

Sub. $x=1$, $-80=C\times 4^2$

$-5=C.$

Sub. $x=-3$, $-16=-4A$

$4=A.$

Sub. $x=0$, $A=4$, $C=-5.$

$-64=-4-3B-45$

$B=5.$

$\therefore \displaystyle\frac{-16x-64}{(x+3)^2(x-1)}=\frac{4}{(x+3)^2}+\frac{5}{(x+3)}-\frac{5}{(x-1)}$

(ii) $\displaystyle\int \frac{-16x-64}{(x+3)^2(x-1)}\,dx$

$$=\int \frac{4}{(x+3)^2}+\frac{5}{(x+3)}-\frac{5}{(x-1)}\,dx$$

$$=-\frac{4}{(x+3)}+5\ln(x+3)-5\ln(x-1)+C.$$

Solutions: GHS Trial p.5.

Q.(12)(c) $\int x\cos x.dx$ $\quad u = x \quad v = \sin x$

$u' = 1. \quad v' = \cos x$

By $\int u v'.dx = uv - \int v u'.dx$

$$\int x\cos x.dx = x\sin x + \int \sin x.dx$$

$$\int x\cos x\,dx = x\sin x + \cos x + C.$$

(d)(i) Moving in SHM if $\ddot{x} = -n^2 x$.

$x = 5\cos 2t - 5\sqrt{3}\sin 2t$

$\dot{x} = -10\sin 2t - 10\sqrt{3}\cos 2t$

$\ddot{x} = -20\cos 2t + 20\sqrt{3}\sin 2t$

$= -4(5\cos 2t - 5\sqrt{3}\sin 2t)$

$\ddot{x} = -4x \Rightarrow$ SHM with $n = 2$.

[Could also find $-4x$ separately & do LHS = RHS].

(ii) $5\cos 2t - 5\sqrt{3}\sin 2t \equiv A\cos 2t\cos\alpha - A\sin 2t\sin\alpha$, $A > 0$

Equating parts,

$5\cos 2t = A\cos 2t\cos\alpha.$	$-5\sqrt{3}\sin 2t = -A\sin 2t\sin\alpha$
$5 = A\cos\alpha$ (1)	$5\sqrt{3} = A\sin\alpha$ (2)

Squaring & adding (1) & (2):

$5^2 + (5\sqrt{3})^2 = A^2(\cos^2\alpha + \sin^2\alpha)$

$10 = A.$

Sub. $A = 10$ in (1)

$5 = 10\cos\alpha \Rightarrow \frac{1}{2} = \cos\alpha.$

Sub. $A = 10$ in (2)

$5\sqrt{3} = 10\sin\alpha \Rightarrow \frac{\sqrt{3}}{2} = \sin\alpha.$

$\alpha = \frac{\pi}{3}.$

$x = 10\cos\left(2t + \frac{\pi}{3}\right)$

(iii) Period of motion $= \frac{2\pi}{2}$ $= \pi$ seconds.	Initial position ($t = 0$) $= 10\cos\frac{\pi}{3}$
Amplitude $= 10$.	$=$ $x = 5$ m POSITIVE (right) of $x = 0$.

Solutions: GHS 4U 2023 Trial p.6

(13)(a)(i) $\begin{pmatrix}-9\\-14\\-3\end{pmatrix}+\lambda_1\begin{pmatrix}3\\2\\2\end{pmatrix}=\begin{pmatrix}23\\-4\\-27\end{pmatrix}+\lambda_2\begin{pmatrix}10\\1\\-16\end{pmatrix}$

Equating x: $-9+3\lambda_1=23+10\lambda_2$

$3\lambda_1-10\lambda_2=32$ (1) ✓

Equating y: $-14+2\lambda_1=-4+\lambda_2$

$2\lambda-\lambda_2=10$ (2) ✓

Equating z: $-3+2\lambda_1=-27-16\lambda_2$

$2\lambda_1+16\lambda_2=-24$ (3)

(3)−(2) $2\lambda_1+16\lambda_2=-24$ (3)

$2\lambda_1-\lambda_2=10$ (2)

$17\lambda_2=-34$

$\underline{\lambda_2=-2.}$

Sub. $\lambda_2=-2$ in (2)

$2\lambda_1+2=10$ ← Be careful. Students must sub. in to an equation already used here.

$\underline{\lambda_1=4.}$

Sub. $\lambda_1=4, \lambda_2=-2$ in (1) to check skewness: $3\times4-10\times-2=32.$ ✓ → MUST do this!

So point of intersection

$=\begin{pmatrix}-9\\-14\\-3\end{pmatrix}+4\begin{pmatrix}3\\2\\2\end{pmatrix}$

$=\underline{\begin{pmatrix}3\\-6\\5\end{pmatrix}}.$

(ii) For 1st line

$x=-9+3\lambda_1, y=-14+2\lambda_1, z=-3+2\lambda_1.$

Sub. in to sphere,

$(x-1)^2+(y+2)^2+(z-4)^2=21.$

$(3\lambda_1-10)^2+(2\lambda_1-12)^2+(2\lambda_1-7)^2=21.$

$17\lambda_1^2-136\lambda_1+293=21.$

$\lambda^2-8\lambda+16=0$ -21 & $\div17$ → Could also show tangent using $\Delta=(-8)^2-4\times1\times16=0.$

$(\lambda_1-4)^2=0 \Rightarrow \underline{\lambda_1=4}.$ Only 1 point of intersection → tangent.

Point of intersection $=\begin{pmatrix}-9\\-14\\-3\end{pmatrix}+4\begin{pmatrix}3\\2\\2\end{pmatrix}=\underline{\begin{pmatrix}3\\-6\\5\end{pmatrix}}.$

Q. (13)(a)(iii) Let direction vector of line $\perp$ to both $= \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$

$$\therefore \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = 0 \quad \& \begin{pmatrix} 10 \\ 1 \\ -16 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = 0$$

$$3x_1 + 2y_1 + 2z_1 = 0 \ (1)$$
$$10x_1 + y_1 - 16z_1 = 0 \ (2) \times 2 = (3)$$
$$3x_1 + 2y_1 + 2z_1 = 0 \ (1) -$$
$$20x_1 + 2y_1 - 32z_1 = 0 \ (3)$$
$$-17x_1 + 34z_1 = 0 \quad \div -17$$
$$x_1 \quad 2z_1 = 0$$
$$x_1 = 2z_1.$$

Sub. in (1): $6z_1 + 2y_1 + 2z_1 = 0$

$$y_1 = -4z_1.$$

Letting $z_1 = 1$, $x_1 = -2$, $y_1 = 2$.

Line $\perp$ to both

$$= \begin{pmatrix} 3 \\ -6 \\ 5 \end{pmatrix} + \lambda_3 \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}.$$

If Line 3 [just found] is radius, then Line 2 is also a tangent as it is $\perp$ to radius.

Finding if $\begin{pmatrix} 3 \\ -6 \\ 5 \end{pmatrix} + \lambda_3 \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}$ passes through $\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$

By x: $3 + 2\lambda_3 = 1$. $\lambda_3 = -1$.

Sub. $\lambda_3 = -1$ in Line 3:

$$\begin{pmatrix} 3 \\ -6 \\ 5 \end{pmatrix} - 1 \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} = \underline{\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}} = \underline{\text{sphere centre.}}$$

As Line 2 is $\perp$ to radius at point of contact on surface of sphere, line 2 $= \begin{pmatrix} 23 \\ -4 \\ -27 \end{pmatrix} + \lambda_2 \begin{pmatrix} 10 \\ 1 \\ -16 \end{pmatrix}$ is ALSO tangent to sphere.

NOTE: Could also use "or otherwise" & show only 1 point of intersection of Line 2 & Sphere.

Q.(13)(b) Step 1: Show true for $n=2$.

LHS $= 7^2 = 49$.

RHS $= 5^2+2^2 = 29$.

LHS $>$ RHS $\Rightarrow$ True for $n=2$.

Step 2: Assume true for $n=k$, $k \geqslant 2$.

i.e. $7^k > 5^k + 2^k$

Step 3: Prove true for $n=k+1$

i.e. RTP: $7^{k+1} > 5^{k+1} + 2^{k+1}$, $k \geqslant 2$.

LHS

$$
\begin{aligned}
&7^{k+1} \\
&= 7 \times 7^k \\
&> 7(5^k+2^k) \text{ [By assumption]} \\
&= (5+2)(5^k+2^k) \\
&= 5^{k+1} + 5 \times 2^k + 2 \times 5^k + 2^{k+1} \\
&> 5^{k+1} + 2^{k+1}
\end{aligned}
$$

LHS $>$ RHS.

If it is true for $n=k$, it will be true for $n=k+1$. Hence it will be true $\forall\ k \in \mathbb{Z}^+ \geqslant 2$.

[Note: Could also be approached by proving $7^{k+1} - (5^{k+1} + 2^{k+1}) > 0$].

(c) Assume $\log_7 19$ is rational

i.e. $\log_7 19 = \frac{p}{q}$, $p, q \in \mathbb{Z}$.

$$
\begin{aligned}
\therefore\ 7^{\frac{p}{q}} &= 19 \\
7^p &= 19^q
\end{aligned}
$$

But 7 & 19 are both prime, so a number would have to exist s.t. its prime factors are 7^p & 19^q.

$\therefore$ This is not possible by the fundamental theorem of arithmetic.

$\therefore$ There is a contradiction $\Rightarrow \log_7 19$ is irrational.

Solutions: GHS 2023 Trial p-9

Q.(14)(a)(i) If a, b real

$(a-b)^2 \geq 0$

$a^2 - 2ab + b^2 \geq 0$

$a^2 + b^2 \geq 2ab.$

(ii) Using (i) $a^2 + b^2 \geq 2ab$ (1)

$a^2 + c^2 \geq 2ac$ (2)

$b^2 + c^2 \geq 2bc$ (3)

(1)+(2)+(3)

$2(a^2+b^2+c^2) \geq 2(ab+ac+bc)$

$a^2+b^2+c^2 \geq ab+ac+bc$

(as required).

(iii) If a, b, c are Δ sides:

$c+b > a$ or $c+a > b$

$c > a-b$ $\quad c > b-a$

$\therefore |c| > |a-b|$

$c^2 > (a-b)^2.$

Note: IMPORTANT! could also use cosine rule:

$c^2 = a^2+b^2 - 2ab\cos C^\circ.$

$> a^2+b^2-2ab$ as $\cos C \leq 1$

& if $\cos C = 1$, no Δ.

(iv) Using (iii) $c^2 > a^2 - 2ab + b^2$ (1)

$a^2 > b^2 - 2bc + c^2$ (2)

$b^2 > a^2 - 2ac + c^2$ (3)

(1)+(2)+(3) $a^2+b^2+c^2 > 2(a^2+b^2+c^2) - 2(ab+ac+bc)$ $+2(ab+ac+bc) - (a^2+b^2+c^2)$

$2ab + 2ac + 2bc > a^2+b^2+c^2$

as required.

Solutions: 2027 GHS 4U Trial p.10

Q.(14)(b)(i) $I_n = \int_0^{\frac{\pi}{4}} \sin^n x \, dx$ $\quad u = \sin^{n-1} x \quad v = -\cos x$

$u' = (n-1)\sin^{n-2} x \cos x \quad v' = \sin x$

By $\int uv' \, dx = uv - \int vu' \, dx$

$$I_n = \int_0^{\frac{\pi}{4}} \sin^n x \, dx = \left[-\cos x \sin^{n-1} x\right]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \sin^{n-2} x \cos^2 x \, dx$$

$$I_n = \left[-\cos\frac{\pi}{4}\left(\sin\frac{\pi}{4}\right)^{n-1} - 0\right] + (n-1)\int_0^{\frac{\pi}{4}} (1-\sin^2 x)\sin^{n-2} x \, dx$$

$$I_n = \left(-\frac{1}{\sqrt{2}}\right) \times \left(\frac{1}{\sqrt{2}}\right)^{n-1} + (n-1)\int_0^{\frac{\pi}{4}} \sin^{n-2} x \, dx - (n-1)\int_0^{\frac{\pi}{4}} \sin^n x \, dx$$

$$I_n = -\left(\frac{1}{\sqrt{2}}\right)^n + (n-1) I_{n-2} - (n-1) I_n.$$

$$n I_n = -\left(\frac{1}{\sqrt{2}}\right)^n + (n-1) I_{n-2}$$

$$I_n = \frac{-\left(\frac{1}{\sqrt{2}}\right)^n}{n} + \frac{n-1}{n} I_{n-2}.$$

(ii) $I_1 = \int_0^{\frac{\pi}{4}} \sin x \, dx$

$$= \left[-\cos x\right]_0^{\frac{\pi}{4}}$$

$$= \left(-\frac{1}{\sqrt{2}} + 1\right)$$

$$= \left(1 - \frac{1}{\sqrt{2}}\right).$$

$$I_3 = \frac{-\left(\frac{1}{\sqrt{2}}\right)^3}{3} + \frac{2}{3}\left(1 - \frac{1}{\sqrt{2}}\right)$$

$$= \frac{2}{3} - \frac{5}{6\sqrt{2}}.$$

$$I_5 = \frac{-\left(\frac{1}{\sqrt{2}}\right)^5}{5} + \frac{4}{5}\left(\frac{2}{3} - \frac{5}{6\sqrt{2}}\right)$$

$$= \frac{8}{15} - \frac{43}{60\sqrt{2}}$$

or $I_5 = \dfrac{64 - 43\sqrt{2}}{120}.$

Solutions: 2023 GHS 4U Trial p.11

Q.(14)(c) $\displaystyle\int\sqrt{\frac{x}{2-x}}\,dx \quad \frac{\times\sqrt{x}}{\times\sqrt{x}}$

$$= \int \frac{x}{\sqrt{2x-x^2}}\,dx$$

$$= \int \frac{x-1}{\sqrt{2x-x^2}}\,dx + \int \frac{1}{\sqrt{1-(x^2-2x+1)}}\,dx$$

$$= -\frac{1}{2}\int \frac{2-2x}{\sqrt{2x-x^2}}\,dx + \int \frac{1}{\sqrt{1-(x-1)^2}}\,dx$$

$$= -\frac{1}{2}\times 2\sqrt{2x-x^2} + \sin^{-1}(x-1) + c.$$

$\left[\text{by } \int f'(x).[f(x)]^n\,dx = \frac{1}{n+1}[f(x)]^{n+1}\right]$

$$\underline{= -\sqrt{2x-x^2} + \sin^{-1}(x-1) + c.}$$

(d)(i) By Euler: $\cos n\theta + i\sin n\theta = e^{ni\theta}$ (1)

$\cos(-n\theta) + i\sin(-n\theta) = e^{-ni\theta}$

$\therefore \cos n\theta - i\sin n\theta = e^{-ni\theta}$ (2)

[as cos even, sin odd].

(1) − (2): $2i\sin n\theta = e^{ni\theta} - e^{-ni\theta}$.

(ii) Hence $[2i\sin\theta]^5$

$= (e^{i\theta} - e^{-i\theta})^5$

$= e^{5i\theta} - 5e^{3i\theta} + 10e^{i\theta} - 10e^{-i\theta} + 5e^{-3i\theta} - e^{-5i\theta}$

$= (e^{5i\theta} - e^{-5i\theta}) - 5(e^{3i\theta} - e^{-3i\theta}) + 10(e^{i\theta} - e^{-i\theta})$

$\therefore 32i\sin^5\theta = 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta \quad \div 32i$

$\sin^5\theta = \dfrac{\sin 5\theta}{16} - \dfrac{5\sin 3\theta}{16} + \dfrac{5\sin\theta}{8}$

as required.

Solutions: 2023 GHS Trial p.12

Q.(15)(a) Centre of motion: $0.2m = C$

(i) Amplitude $= 0.05m$.

Period $= \frac{1}{100}$ second.

$\therefore \frac{2\pi}{n} = \frac{1}{100}$ $\times 100n$ $\times 100n$

$\underline{200\pi = n.}$

$\therefore \ddot{x} = -(200\pi)^2(x - 0.2).$

Maximum force will happen at max [or min] accel

i.e. where $x = 0.15$ or $x = 0.25$

[Force will be same, accel $= \pm$].

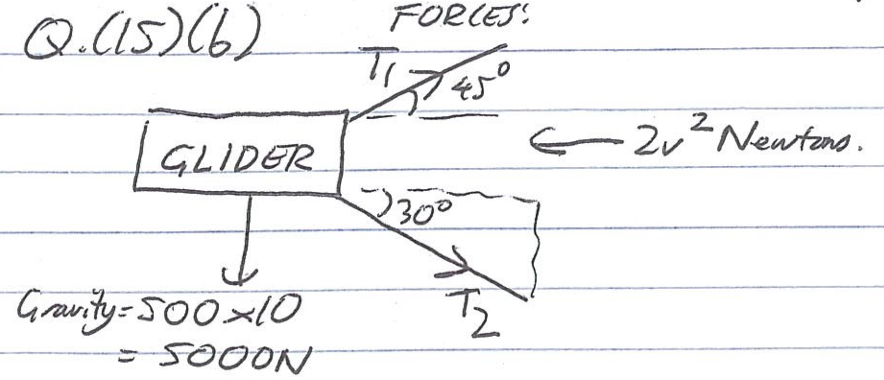
At $x = 0.15$, $a = -(200\pi)^2(0.15 - 0.2)$.

By $F = ma$,

$F = -(200\pi)^2(-0.05) \times 1.2$

$= 23\ 687$ Newtons [nearest Newton].

Q.(15)(b)



(i) Resolving forces vertically

$$T_1 \sin 45^\circ = T_2 \sin 30^\circ + 5000$$

$$\frac{T_1\sqrt{2}}{2} = \frac{T_2}{2} + 5000$$

$$T_1\sqrt{2} = T_2 + 10\,000$$

$$T_1\sqrt{2} - 10000 = T_2 \quad \text{as required.}$$

(ii) Resolving horizontally:

$$F = T_1 \cos 45^\circ + T_2 \cos 30^\circ - 2v^2.$$

$$500a = \frac{10000 \times \sqrt{2}}{2} + \frac{(10\,000\sqrt{2} - 10\,000) \times \sqrt{3}}{2} - 2v^2$$

$$500a = 5000\sqrt{2} + 5000\sqrt{6} - 5000\sqrt{3} - 2v^2. \quad \div 500$$

$$a = \ddot{x} = 10(\sqrt{2} + \sqrt{6} - \sqrt{3}) - \frac{v^2}{250}$$

Limiting speed is where $\ddot{x} \to 0$

i.e. $\frac{v^2}{250} \to 10(\sqrt{2} + \sqrt{6} - \sqrt{3})$

Limiting $v = 73$m/s.

(iii) $a = v \cdot \frac{dv}{dx} = 10(\sqrt{2} + \sqrt{6} - \sqrt{3}) - \frac{v^2}{250}$

$$\frac{dv}{dx} = \frac{2500(\sqrt{2} + \sqrt{6} - \sqrt{3}) - v^2}{250v}$$

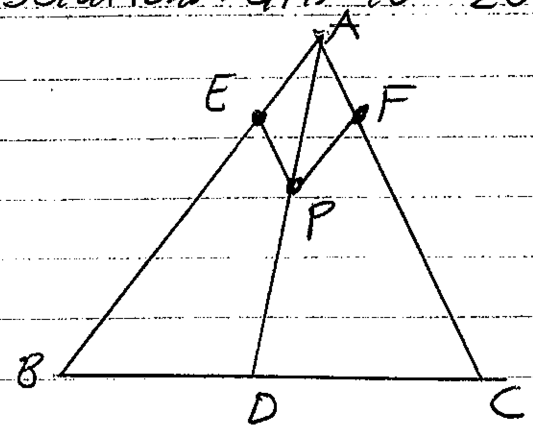
$$\frac{dx}{dv} = \frac{250v}{2500(\sqrt{2} + \sqrt{6} - \sqrt{3}) - v^2}$$

$$x = \int_{50}^{70} \frac{-2v}{2500(\sqrt{2} + \sqrt{6} - \sqrt{3}) - v^2} \, dv$$

$$= \left[-125\ln\left[2500(\sqrt{2} + \sqrt{6} - \sqrt{3}) - v^2\right]\right]_{50}^{70}$$

$x = 235 \cdot 7$m. The glider takes 235·7m to get to 70m/s.

Q.(15)(c)



(i) $\overrightarrow{BD} = \lambda \overrightarrow{BC}$

$= \lambda(\overrightarrow{BA} + \overrightarrow{AC})$

$= \lambda(-\underset{\sim}{p} + \underset{\sim}{q})$

$= \lambda(\underset{\sim}{q} - \underset{\sim}{p})$.

$\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD}$

$= \underset{\sim}{p} + \lambda(\underset{\sim}{q} - \underset{\sim}{p})$

(ii) Letting $\overrightarrow{AP} = k\overrightarrow{AD}$

$\overrightarrow{AP} = k[\underset{\sim}{p} + \lambda\underset{\sim}{q} - \lambda\underset{\sim}{p}]$

$\overrightarrow{AP} = (k - k\lambda)\underset{\sim}{p} + k\lambda\underset{\sim}{q}$ (1)

As $\overrightarrow{AP} = \overrightarrow{AE} + \overrightarrow{EP}$

$\overrightarrow{AP} = \frac{1}{3}\underset{\sim}{p} + \frac{1}{5}\underset{\sim}{q}$. (2)

$\therefore$ Equating (1) & (2):

$\frac{1}{3}\underset{\sim}{p} + \frac{1}{5}\underset{\sim}{q} = (k - k\lambda)\underset{\sim}{p} + k\lambda\underset{\sim}{q}$

$\therefore k - k\lambda = \frac{1}{3}$ & $k\lambda = \frac{1}{5}$.

$\therefore k - \frac{1}{5} = \frac{1}{3} \Rightarrow k = \frac{8}{15}$.

As $k\lambda = \frac{1}{5}$, $\frac{8}{15}\lambda = \frac{1}{5}$

$\lambda = \frac{1}{5} \times \frac{15}{8}$

$= \frac{3}{8}$.

$\therefore \lambda = \frac{3}{8}$ & $\overrightarrow{BD} = \frac{3}{8}\overrightarrow{BC}$.

Q.(16)(a) By DeMoivre's theorem,

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

$\therefore \cos 7\theta + i\sin 7\theta = (\cos\theta + i\sin\theta)^7$

$= \cos^7\theta + 7i\cos^6\theta\sin\theta - 21\cos^5\theta\sin^2\theta - 35i\cos^4\theta\sin^3\theta$
$+ 35\cos^3\theta\sin^4\theta + 21i\cos^2\theta\sin^5\theta - 7\cos\theta\sin^6\theta - i\sin^7\theta$.

Equating imaginaries

$\sin 7\theta = 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta$

$= 7(1-\sin^2\theta)^3\sin\theta - 35(1-\sin^2\theta)^2\sin^3\theta + 21(1-\sin^2\theta)\sin^5\theta$

$= 7\sin\theta(1 - 3\sin^2\theta + 3\sin^4\theta - \sin^6\theta) - 35(1 - 2\sin^2\theta + \sin^4\theta)\sin^3\theta$
$+ 21\sin^5\theta(1-\sin^2\theta) - \sin^7\theta$

$= 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$

as required.

(ii) $-\sin 7\theta = 64\sin^7\theta - 112\sin^5\theta + 56\sin^3\theta - 7\sin\theta$

So if $64x^7 - 112x^5 + 56x^3 - 7x + 1 = 0$

$64x^7 - 112x^5 + 56x^3 - 7x = -1$

$-\sin 7\theta = -1$

& $\sin 7\theta = 1$

where $x = \sin\theta$.

$\therefore \sin 7\theta = 1$

$7\theta = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{13\pi}{2}, \frac{17\pi}{2}, \frac{21\pi}{2}, \frac{25\pi}{2}$

$\theta = \frac{\pi}{14}, \frac{5\pi}{14}, \frac{9\pi}{14}, \frac{13\pi}{14}, \frac{17\pi}{14}, \frac{3\pi}{2}, \frac{25\pi}{14}$.

Solutions to $64x^7 - 112x^5 + 56x^3 - 7x + 1 = 0$

are -1 $(\sin\frac{3\pi}{2})$, $\sin\frac{\pi}{14}$, $\sin\frac{5\pi}{14}$, $\sin\frac{9\pi}{14}$, $\sin\frac{13\pi}{14}$, $\sin\frac{17\pi}{14}$, $\sin\frac{25\pi}{14}$.

Q. (16)(a)(iii)

By Product of roots of

$$64x^7 - 112x^5 + 56x^3 - 7x + 1 = 0$$

$$-1 \times \sin\frac{\pi}{14} \times \sin\frac{5\pi}{14} \times \sin\frac{9\pi}{14} \times \sin\frac{13\pi}{14} \times \sin\frac{17\pi}{14} \times \sin\frac{25\pi}{14} = -\frac{1}{64}.$$

$\div -1$ & noting $\sin\frac{5\pi}{14} = \sin\frac{9\pi}{14}$, $\sin\frac{\pi}{14} = \sin\frac{13\pi}{14}$

& $\sin\frac{17\pi}{14} = \sin\frac{25\pi}{14}$.

$$\sin^2\frac{\pi}{14} \times \sin^2\frac{5\pi}{14} \times \sin^2\frac{17\pi}{14} = \frac{1}{64}. \quad \sqrt{\ }$$

$$\sin\frac{\pi}{14} \times \sin\frac{5\pi}{14} \times \sin\frac{17\pi}{14} = \pm\frac{1}{8}$$

But as $\sin\frac{\pi}{14}$, $\sin\frac{5\pi}{14}$ positive &

$\sin\frac{17\pi}{14}$ negative,

answer is negative.

$$\sin\frac{\pi}{14} \times \sin\frac{5\pi}{14} \times \sin\frac{17\pi}{14} = -\frac{1}{8}.$$

Note: There are a LOT of "or otherwise" methods here!).

Solutions: GHS 4U 2023 Trial p.17

Q.(16)(b)(i)

$0{\cdot}2Mv \quad 10M$

$$F = Ma = -0{\cdot}2Mv - 10M$$

$$\therefore a = \frac{dv}{dt} = -0{\cdot}2v - 10$$

$$\frac{dt}{dv} = -\frac{1}{0{\cdot}2v+10}$$

$$= \frac{-5}{v+50}.$$

$$t = \int_{v_0}^{v} \frac{-5}{v+50}\,.dv$$

Note: Could also do $t = \int \frac{-5}{v+50}\,.dt = -5\ln(v+50)+c$ & use $v = v_0$ when $t=0$.

$$= \left[-5\ln(v+50)\right]_{v_0}^{v}$$

$$= -5\ln(v+50) + 5\ln(v_0+50)$$

$$t = -5\ln\left[\frac{v+50}{v_0+50}\right].$$

Time to max. height $v = 0$

$$t = -5\ln\left(\frac{50}{v_0+50}\right)$$

Time to max. height. $= 5\ln\left(\frac{v_0+50}{50}\right).$

(ii) $t = -5\ln\left[\frac{v+50}{v_0+50}\right].$

$$\therefore e^{-0{\cdot}2t} = \frac{v+50}{v_0+50}$$

$$(v_0+50)e^{-0{\cdot}2t} - 50 = v.$$

$$x = \int_0^t (v_0+50)e^{-0{\cdot}2t} - 50\,.dt$$

Could also do $\Rightarrow x = \int (v+50)e^{-0{\cdot}2t} - 50\,.dt$ & use $x = 0$ when $t=0$.

$$= \left[-5(v_0+50)e^{-0{\cdot}2t} - 50t\right]_0^t$$

$$= \left[-5[v_0+50]e^{-0{\cdot}2t} - 50t\right] + 5[v_0+50]e^{-0{\cdot}2\times 0} + 50\times 0$$

$$x = -5[v_0+50]e^{-0{\cdot}2t} - 50t + 5[v_0+50].$$

Q.(16)(b)(ii) (continued):

Max. height when $v=0$ [so $t = 5\ln\left[\frac{v_0+50}{50}\right]$ from (i)].

Note: $-0.2t = \ln\left[\frac{50}{v_0+50}\right]$

& $e^{-0.2t} = \frac{50}{v_0+50}$.

So max. height $= -5[v_0+50]\times\frac{50}{v_0+50} - 50\times 5\ln\left[\frac{v_0+50}{50}\right] + 5[v_0+50]$

$= -250 - 250\ln\left[\frac{v_0+50}{50}\right] + 5v_0 + 250$

Max. height reached $= 5v_0 - 250\ln\left(\frac{v_0+50}{50}\right)$.

(16)(b)(iii)

Projectile falling from rest:

$\downarrow$
$10M$

$0{\cdot}2Mv$
$\uparrow$

$$F = Ma = 10M - 0{\cdot}2MV$$

$$a = \frac{dv}{dt} = 10 - 0{\cdot}2v$$

$$\frac{dt}{dv} = \frac{1}{10 - 0{\cdot}2v}$$

$$= \frac{5}{50 - v}.$$

$$t = -5\int_0^v \frac{-1}{50-v}\,dv$$

$$= -5\Big[\ln(50-v)\Big]_0^v$$

$$= -5\ln\left[\frac{50-v}{50}\right].$$

$$e^{-0{\cdot}2t} = \frac{50-v}{50}$$

$$v = 50 - 50e^{-0{\cdot}2t}.$$

$$x = \int_0^t 50 - 50e^{-0{\cdot}2t}\,dt$$

$$= \Big[50t + 250e^{-0{\cdot}2t}\Big]_0^t$$

$$= 50t + 250e^{-0{\cdot}2t} - 250.$$

(16)(b)(iv) From (i) & (ii):

As projectile reached a height of $5v_0 - 250\ln\left[\frac{v_0+50}{50}\right]$

after $5\ln\left[\frac{v_0+50}{50}\right]$ seconds,

it will also fall $5v_0 - 250\ln\left[\frac{v_0+50}{50}\right]$ (1)

in $10 - 5\ln\left[\frac{v_0+50}{50}\right]$ seconds.

Using x when projectile is falling

$x = 50t + 250e^{-0.2t} - 250$

Note: If $t = 10 - 5\ln\left[\frac{v_0+50}{50}\right]$

$t = \ln[e^{10}] - 5\ln\left[\frac{v_0+50}{50}\right]$

$= -5\ln[e^{-2}] - 5\ln\left[\frac{v_0+50}{50}\right]$

$= -5\ln\left[\frac{v_0+50}{50e^2}\right]$.

$e^{-0.2t} = \frac{v_0+50}{50e^2}$.

& $x = 50t + 250e^{-0.2t} - 250$

$= 50 \times \left[10 - 5\ln\left[\frac{v_0+50}{50}\right]\right] + 250 \times \frac{(v_0+50)}{50e^2} - 250$

$= 500 - 250\ln\left[\frac{v_0+50}{50}\right] + \frac{5(v_0+50)}{e^2} - 250$

$x = 250 + \frac{5(v_0+50)}{e^2} - 250\ln\left(\frac{v_0+50}{50}\right)$ (2)

As this is how far projectile has fallen, (1) = (2)

$5v_0 - 250\ln\left[\frac{v_0+50}{50}\right] = 250 + \frac{5(v_0+50)}{e^2} - 250\ln\left[\frac{v_0+50}{50}\right]$.

$5v_0 = 250 + \frac{5(v_0+50)}{e^2}$

$5e^2 v_0 = 250e^2 + 5v_0 + 250$

$(5e^2 - 5)v_0 = 250e^2 + 250$

$v_0 = 65.65\ldots$ m/s.